* EIGEN VALUES AND EIGEN VECTORS

If A is Square Matrix of order n, we can form Homogeneous System. $\boxed{(A - \lambda I) X = 0}$ i.e. $AX = \lambda X$

The determinant of $[A - \lambda I]$ Matrix i.e.

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$ is called Characteristic equation of A.

So $|A - \lambda I| = 0$ is Characteristic equation of A.

On expand the determinant, the Ch. equation can be written as.

$$\boxed{\lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \cdots + k_n = 0} \rightarrow \text{(Ch. Polynomial)}$$

if 2×2 matrix $\rightarrow$ we will get Quadratic Equation.
3×3 Matrix $\rightarrow$ we will get Cubic equation.

The Roots of Ch. polynomial are called $\boxed{\text{Eigen Values}}$, latent Values, or characteristic roots.

$\boxed{\text{Now Solve } (A - \lambda I) X = 0}$ it will always give $\boxed{\text{non-trivial Solution}}$ $\boxed{\text{corresponding to each Eigen Value.}}$

It is again homogeneous System and find $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, which is $\boxed{\text{Eigen Vector or latent Vector}}$ corresponding to that eigen Value.

NOTE: Eigen vector corresponding to an eigen Value is not unique.

* TRACE of Square Matrix = Sum of its diagonal elements
= Sum of eigen values.

Ex-1 Find Eigen value of the Matrix $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

Solu. The Characteristic equation $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - (-5)(-2)$$
$$= 4 - \lambda - 4\lambda + \lambda^2 - 10 = \lambda^2 - 5\lambda - 6 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda - 6 = 0 \Rightarrow \lambda^2 - 6\lambda + \lambda - 6 = 0 \Rightarrow \boxed{\lambda = 6, -1}.$$

Ex-2 Find Eigen Value and Eigen Vector of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$(A - \lambda I)X = 0$

Solu. The characteristic Eqn. will be $|A - \lambda I| = 0$

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} \overset{+}{1-\lambda} & \overset{-}{1} & \overset{+}{3} \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}$$

$$\Rightarrow (1-\lambda)[(5-\lambda)(1-\lambda) - 1] - 1[1 - \lambda - 3] + 3[1 - 3(5-\lambda)] = 0$$
$$\Rightarrow (1-\lambda)[5 - 5\lambda - \lambda + \lambda^2 - 1] - 1 + \lambda + 3 + 3(1 - 15 + 3\lambda) = 0$$
$$\Rightarrow (1-\lambda)[\lambda^2 - 6\lambda + 4] + \lambda + 2 - 42 + 9\lambda = 0$$
$$\Rightarrow \lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + \lambda + 2 - 42 + 9\lambda = 0$$
$$\Rightarrow -\lambda^3 + 7\lambda^2 - \cancel{10\lambda} + \cancel{10\lambda} - 36 = 0$$
$$\Rightarrow \boxed{\lambda^3 - 7\lambda^2 + 36 = 0}$$ → ? How to solve. Make or Take Multiples of 36. which satisfies the equation. (any one factor or root.

Here -2 satisfies the equation.

$$(-2)^3 - 7(-2)^2 + 36 = -8 - 28 + 36$$
$$= -36 + 36 = 0.$$

+1, −1, 36, −36
+2, −2, 18, −18
3, −3, 12, −12.

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & 0 \end{array}$$

So $\boxed{\lambda = -2, 3, 6}$

$\lambda^2 - 9\lambda + 18 = 0 \Rightarrow$

$\lambda^2 - 6\lambda - 3\lambda + 18 = 0$

$\Rightarrow \boxed{\lambda = 3, 6}$

Now find eigen Vectors corresponding to them.

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Now for $\boxed{\lambda = -2}$ $\Rightarrow \begin{bmatrix} 1+2 & 1 & 3 \\ 1 & 5+2 & 1 \\ 3 & 1 & 1+2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

operate R_{12}, then $\begin{bmatrix} 1 & 7 & 1 \\ 3 & 1 & 3 \\ 3 & 1 & 3 \end{bmatrix}$

Operate $R_2 \to R_2 - 3R_1$ & $R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & -20 & 0 \end{bmatrix}$, operate $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 7y + z = 0$
$-20y = 0$, So $y = 0$, $\boxed{x = -z}$

Now take $x = k$, so $z = -k$, $y = 0$, so $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Now for $\boxed{\lambda = 3}$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate R_{12}

$\sim \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix} \Rightarrow$ Operate $R_2 + 2R_1$, $R_3 - 3R_1$ $\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \xrightarrow[R_3 + R_2]{\text{operate}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x + 2y + z = 0$
$5y + 5z = 0$
$\Rightarrow \boxed{y = -z}$

$\Rightarrow x = -2y - z$
$= -2y + y$
$\boxed{x = -y}$

, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ y \\ -y \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

let $\boxed{y = k}$

Now for $\boxed{\lambda = 6}$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So operate R_{12} $\sim \begin{bmatrix} 1 & -1 & 1 \\ -5 & 1 & 3 \\ 3 & 1 & -5 \end{bmatrix}$

operate $R_2 \to R_2 + 5R_1$ & $R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix}$ operate $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\Rightarrow x - y + z = 0$

$-4y + 8z = 0$ $\Rightarrow \boxed{y = 2z}$ $\Rightarrow x = y - z = 2z - z = z$

So $x = z$

$y = 2z$

$z = z$

let $z = k$ so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$

So eigen vectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ +1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Modal Matrix $= P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & +1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, Find $P^{-1} = \dfrac{\operatorname{adj} P}{|P|} = -\dfrac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix}$

$\therefore$ Diagonal form $D = P^{-1} A P$

$$= \frac{-1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & (-1) & 1 \\ 0 & (+1) & 2 \\ 1 & (-1) & 1 \end{bmatrix}$$

$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ which is formed by eigen values of A.

(*) Find Eigen value & Eigen Vector of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Soln. $(A - \lambda I)X = 0$

The ch. equation will be $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} (-2-\lambda) & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$$\Rightarrow (-2-\lambda)\left[(1-\lambda)(-\lambda) - (-6)(-2)\right] - 2\left[-2\lambda - (-1)(-6)\right] - 3\left\{-4 - (-1)(1-\lambda)\right\} = 0$$

$$\Rightarrow (-2-\lambda)\{-\lambda + \lambda^2 - 12\} - 2\{-2\lambda - 6\} - 3\{-4 + 1 - \lambda\} = 0$$

$$\Rightarrow \boxed{\lambda^3 + \lambda^2 - 21\lambda - 45 = 0}$$

Here $\boxed{\lambda = -3}$ Satisfies the equation (hit and trial)

$$\begin{array}{r|rrr|r} -3 & 1 & 1 & -21 & -45 \\ & & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

$\lambda^2 - 2\lambda - 15 = 0$

So $\boxed{\lambda = -3, 5}$

Here $\boxed{\lambda = -3, -3, 5} \rightarrow$ Eigen Values.

Now $\boxed{\lambda = 5}$,

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate R_{13}

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \xrightarrow[R_3 - 7R_1]{\text{Operate } R_2 + 2R_1} \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix}$$

Operate $R_3 \rightarrow R_3 + 2R_2$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x - 2y - 5z = 0$

$-8y - 16z = 0 \Rightarrow \boxed{y = -2z}$

Put in first equation

$-x + 4z - 5z = 0$

$\Rightarrow -x - z = 0 \Rightarrow \boxed{x = -z}$

So let $z = k$, so

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = -k\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Corresponding to $\lambda = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

operate $R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x + 2y - 3z = 0$

Put $y = 0$, $z = 1$, $x = +3z$

So $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Put $z = 0$, $y = 1$, $x = -2y$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

$\therefore$ Eigen vector corresponding to $\lambda = -3$ are.

$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

Here Modal Matrix $P = \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$, find $P^{-1} = \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}$

Diagonal Matrix $D = P^{-1}AP$

$$= \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & -6 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

i.e. the diagonal Matrix formed by eigenValues $\lambda = -3, -3, 5$.

[TRY] (1) Find Eigen Value and Eigen Vector.

$$A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix},$$

Ch. eq. $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1, 2, 3$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(2) Find eigen Value and eigen Vector

$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

$\boxed{\lambda = -1, 3, 4}$

$X_1 = \begin{bmatrix} -3 \\ 1 \\ 6 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}, X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$